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The constructal law and the thermodynamics of flow systems with configuration

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Abstract

In this paper we develop an analytical and graphical formulation of the constructal law of maximization of flow access in systems with heat and fluid flow irreversibilities and freedom to change configuration. The flow system has global performance (e.g., minimization of global flow resistance) and global properties, or constraints (e.g., overall size, and total duct volume). The infinity of possible flow structures occupies a region of the two-dimensional domain of ''global performance versus freedom to morph''. This region of ''nonequilibrium flow structures'' is bounded by a line representing the best flow structures that are possible when the freedom to morph is limited. The best of all such structures are the ''equilibrium flow structures'': here the performance level is the highest, and it does not change even though the flow architecture can change with maximum freedom. The universality of this graphical and analytical presentation is illustrated with examples of flow structures from three classes: flow between two points, flow between a circle and its center, and flow between one point and an area. In sum, this paper presents an analytical and graphical formulation of the constructal principle of generation of flow architecture. The place of this new and self-standing principle in the greater framework of thermodynamics is outlined.

Keywords: Constructal theory; Design as science; Thermodynamics; Nonequilibrium; Configuration; Survival; Evolution; Territory; Growth

1. Introduction

Flow systems are imperfect thermodynamically because of the resistances that their flows must overcome. Depending on system purpose and complexity, the currents may carry fluids, heat, electricity and chemical species. The resistances are an integral and unavoidable presence because of the finite-size constraints that define the flow system. For example, the resistance to the flow of heat between two streams in a balanced counterflow heat exchanger can be made vanishingly small if the heat transfer surface can be made infinitely large. In reality, the surface size is fixed, and this means that the heat current is destined to encounter a thermal resistance. The current flows irreversibly, and this feature has a negative effect on global thermodynamic performance. The flow system is destined to be imperfect.

When the flow system is complex, the currents and resistances are many and diverse. The route to higher global performance consists of balancing each resistance against the rest. The distributing and re-distributing of imperfection through the complex flow system is accomplished by making changes in the flow architecture. A prerequisite then is for the flow system to be free to change its configuration—free to morph. The morphing of structure is the result of the collision between the global objective and the global constraints.

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Nomenclature

A	area, m^2
A_p	projected area of high-conductivity inserts, m^2
C	constant, Eq. (2)
d	spacing between outlets on the circle, Fig. 3
D_h	hydraulic diameter, m
D_0, D_1, D_2	duct diameters, m
f	friction factor, Eq. (2)
$\hat{f}$	global flow resistance, Eq. (3)
F	Helmholtz free energy, J
k	pairing level
k_0	low thermal conductivity, $W\,m^{-1}\,K^{-1}$
k_p	high thermal conductivity, $W\,m^{-1}\,K^{-1}$
L	system size, m
L_0, L_1, L_2	duct lengths, m
$\dot{m}$	mass flow rate, $kg\,s^{-1}$
n	number of sides in regular polygon
n_0	number of tubes that reach the center of the disc, Fig. 3
N	number of outlets on the circle, Fig. 3
p	perimeter of duct cross-section, m
q'''	volumetric heat generation rate, $W\,m^{-3}$
R	global flow resistance, e.g., $\Delta P/\dot{m}$
R_t	global thermal resistance, $K\,W^{-1}$
Re	Reynolds number
S	entropy, $J\,K^{-1}$
T_{max}	hot-spot temperature, K
T_{min}	heat sink temperature, K
U	internal energy, J
$\overline{U}$	mean velocity, $m\,s^{-1}$
V	volume of all the ducts, m^3
Vol	volume of thermodynamic system, m^3
Y_L, Y_V	first derivatives, Eq. (13)

Greek symbols

ΔP	pressure drop, Pa
ν	kinematic viscosity, $m^2\,s^{-1}$
ρ	density, $kg\,m^{-3}$

The generation of flow architecture is the means by which the flow system achieves its global objective under the constraints.

In recent years, this activity of thermodynamic optimization through the selection of flow configuration has become more focused on the end result, which is the generation of the architecture of the flow system. This is particularly evident in modern computational heat and fluid flow, where large numbers of flow configurations can be simulated, compared and optimized. The generation of flow architecture is a phenomenon at work everywhere, not only in engineered flow systems but also in natural flow systems (animate and inanimate). The universality of this observation was expressed in a compact statement (the constructal law [1,2]) that proclaims a natural tendency in time: the maximization of access for the currents that flow through a morphing flow system. The thought that this principle can be used to rationalize the occurrence of optimized flow structures in nature (e.g., tree networks, round tubes) was named constructal theory [1–4].

This paper is a formulation of the constructal principle in analytical and graphical terms that are analogous to terms employed in thermodynamics [2]. This formulation makes the universality of the constructal law more evident.

2. Flow between two points

A flow system, or nonequilibrium thermodynamic system, is characterized by "properties" (constraints), such as total volume, total volume occupied by all the ducts, etc. A flow system is also characterized by "performance" (function, objective) and "flow structure" (configuration, layout, geometry, architecture). Unlike the black box of classical thermodynamics, which represents a system at equilibrium, a flow system has performance and especially configuration. Each flow system has a *drawing*. By means of examples, we show that each flow system has a fundamental relation between performance, properties and flow structure (drawing).

We start with one of the simplest examples of how the collision between global objective and global constraints generates the complete architecture of the flow system. Consider the flow between two points (Fig. 1), where 'simple' are only the optimal and near-optimal architectures. This makes the example easy to present graphically. The rest of the design process is conceptually as vast and complicated as in any other example. When the flow architecture is free to morph, the design space is infinite. There is an infinity of flow architectures that can be chosen to guide a fluid stream ($\dot{m}$) from one point to another point.

Constructal theory [1,2] begins with the global objective(s) and the global constraint(s) of the flow system, and the fact that in the beginning the geometry of the flow is missing. Geometry is the unknown. In Fig. 1 the objective is to force the single-phase fluid stream $\dot{m}$ to flow from one point to another, while using minimal pumping power. When $\dot{m}$ is fixed, this objective is the same as seeking flow architectures with minimal pressure overall difference (ΔP), minimal overall flow resistance

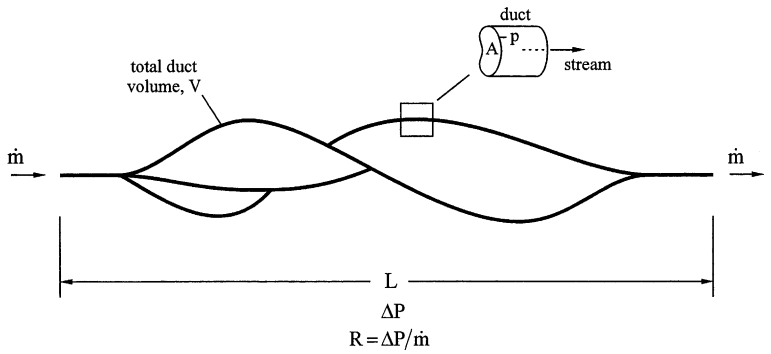

Fig. 1. General, undefined flow architecture for guiding a stream from one point to another point.

(R), or minimal rate of entropy generation by fluid friction.

There are two global constraints, one external and the other internal. The external constraint is the "system size", which is represented by the distance between the two points, L. The internal global constraint is the "amount" invested in making the flow architecture. In Fig. 1 that amount is the total volume (V) of all the ducts of the flow structure. Without such an investment there is no flow—not even a drawing that would show the flow. A flow must be guided. Flow means direction, geometry and architecture, in addition to flow rate.

Why is there an infinity of eligible flow architectures that meet the global objective and global constraints recognized above? There are many reasons, i.e., many thoughts in the direction of which the number of possible architectures increases without bounds:

(i) the flow pattern may be two-dimensional (in the plane of Fig. 1), or three-dimensional,
(ii) any number of ducts may be connected in parallel between the two points,
(iii) a duct may have any number of branches or tributaries at any location between the two points,
(iv) a single duct may have any length,
(v) the cross-sectional shape may vary along the duct,
(vi) a duct may have any cross-sectional shape.

We will see that the best configuration (the straight round tube) is far from being alone on the podium of maximal performance. This podium and the configuration world under it are the new physics domain charted by this paper, and made a part of thermodynamics.

How do we identify the geometric features that bring a flow architecture to the highest level of global performance? There are many lessons of this type throughout engineering, and, if remembered, they constitute *strategy*—they shorten dramatically the search for the geometry in which all the features are "useful" in serving the global objective. Constructal theory is about strategy, about compact lessons of optimal shape and structure, which are fundamental and universally applicable. They are geometric relatives of truths such as the universal observation that all things flow naturally from high to low (the second law of thermodynamics).

Here are the classical lessons that abbreviate the search through the broad categories listed as (i)–(vi). Again, for simplicity assume that ducts are slender, and the flows are slow so that in each cross-section the regime is laminar and fully developed. Each lesson is identified by the symbol of the geometric feature that it addresses:

(i)–(iii) A single duct with large cross-section offers a smaller flow resistance than two ducts with smaller cross-sections connected in parallel.
(iv) The lowest resistance belongs to the shortest duct, in this case the straight duct between the two points.
(v) The duct with cross-sectional geometry that does not vary longitudinally has a lower resistance than the duct with variable cross-section. In heat conduction, for example, this principle is invoked to deduce the optimal fin shape [5]. If the cross-section is the throttle that represents the imperfection (resistance) of the flow path, then, in line with constructal theory, the duct with constant cross-section is the duct with "optimal distribution of imperfection" [1].

Summing up, out of the infinity of designs represented by (i)–(v) we have selected a single straight duct with a cross-sectional shape that does not vary from one end of the duct to the other. According to (vi), however, there is still an infinite number of possible cross-section shapes: symmetric versus asymmetric, smooth versus polygonal, etc. Which impedes the flow the least?

The answer becomes visible if we assume cross-sections with polygonal shapes. Start with an arbitrary cross-section shaped as a triangle. The area of the cross-section A is fixed because the total duct volume V and the duct length L are fixed, namely $A = V/L$. Triangular cross-sections constrict the flow when one of the angles is much smaller than the other two.

The least resistance is offered by the most "open" triangular cross-section, which is shaped as an equilateral triangle. Once again, if one very small angle and two larger ones represent a nonuniform distribution of geometric features of imperfection (i.e., features that impede the flow), then the equilateral triangle represents the architecture with "optimal distribution of imperfection".

The same holds for any other polygonal shape. The least resistance is offered by a cross-section shaped as a regular polygon. In conclusion, out of the infinity of flow architectures recognized in class (vi) we have selected an infinite number of candidates. They are ordered according to the number of sides (n) of the regular polygon, from the equilateral triangle ($n = 3$) to the circle ($n = \infty$). The flow resistance for Hagen–Poiseuille flow through a straight duct with polygonal cross-section can be written as (Ref. [1, pp. 127–128])

$$\frac{\Delta P}{\dot{m}} = \frac{\nu L}{8V^2}\frac{Cp^2}{A} \tag{1}$$

where p is the perimeter of the cross-section. As shown in Table 1, the dimensionless perimeter $p/A^{1/2}$ is only a function of n. The same is true about C, which appears in the solution for friction factor in Hagen–Poiseuille flow,

$$f = \frac{C}{Re} \tag{2}$$

where $Re = \overline{U}D_h/\nu$, $D_h = 4A/p$ and $\overline{U} = \dot{m}/(\rho A)$.

In conclusion, the group Cp^2/A depends only on n, and accounts for how this last geometric degree of freedom influences global performance. The group Cp^2/A is the dimensionless global flow resistance of the flow system. The smallest Cp^2/A value is the best, and the best is the round cross-section.

Table 1
The laminar flow resistances of straight ducts with regular-polygonal cross-sections with n sides

n	C	$p/A^{1/2}$	Cp^2/A
3	40/3	4.559	277.1
4	14.23	4	227.6
5	14.74	3.812	214.1
6	15.054	3.722	208.6
8	15.412	3.641	204.3
10	15.60	3.605	202.7
∞	16	$2\pi^{1/2}$	201.1

3. Equilibrium flow structures

Fig. 2 is a plot of the flow resistance data of Table 1. The flow structure with minimal global resistance is approached gradually (with diminishing decrements) as n increases. The polygonal cross-section with $n = 10$ performs nearly as well as the round cross-section ($n = \infty$). The 'evolution' of the cross-sectional shape stops when the number of features (n) has become infinite, i.e., when the structure has become the most free.

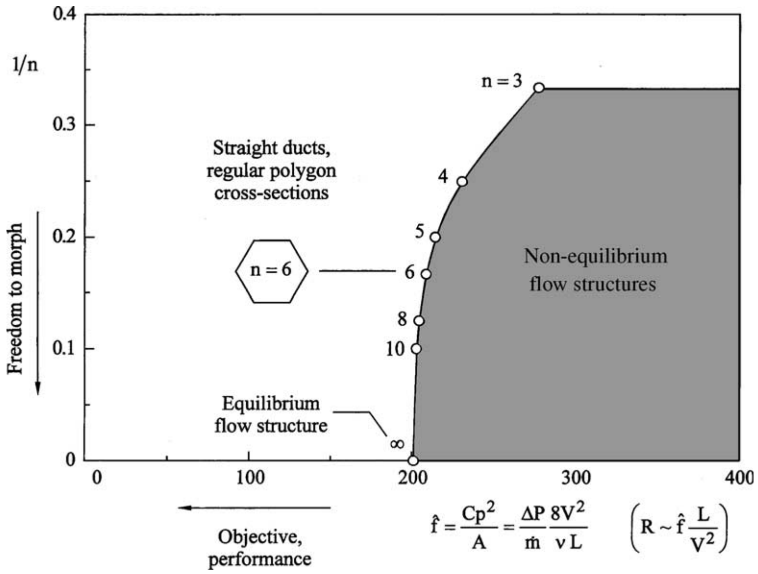

Fig. 2. The approach to the minimal global flow resistance between two points when the number of sides of the regular-polygon cross-section (n) increases (data from Table 1).

This configuration where changes in global performance have stopped is the equilibrium flow architecture.

The curve plotted in Fig. 2 was generated by calculations for regular-polygon cross-sections. The curve is in reality a sequence of discrete points, one point for each n value. We drew a continuous line through these points in order to stress an additional idea. Regardless of n, the regular polygon and straight duct with constant cross-section is already the 'winner' from an infinitely larger group of competing architectures. See features (i)–(vi) and Fig. 1. Qualitatively, this means that the global flow resistances of all the designs that are not covered by Table 1 fall to the right of the curve plotted in Fig. 2.

In sum, the immensely large world of possible designs occupies only a portion of the two-dimensional domain illustrated in Fig. 2. This domain can be described qualitatively as "performance versus freedom", when global properties such as L and V are specified. The boundary of the domain is formed by a collection of the better flow structures. The best is achieved by putting more freedom in the geometry of the flow structure (e.g., a larger n). The best performance belongs to the structure that was most free to morph—the equilibrium configuration. In its immediate vicinity, however, we find many configurations that are different (they have finite n values), but have practically the same global performance level. These are near-equilibrium flow structures.

4. Flow between one point and a large number of points

The evolution of flow configuration illustrated in Figs. 1 and 2 for point-to-point flows is a universal phenomenon, which manifests itself during any search for optimal flow architectures. Some of the more complex architectures that have been optimized recently are the flow structures that connect one point (source, or sink) with an infinity of points (line, area, or volume). According to constructal theory, the best flow path that makes such a connection is shaped as a tree [1–4]. The tree is for point–area flows what the straight duct is for point–point flows.

The search for the flow path with minimal global resistance between one point and an area begins with recognizing the many ways in which the freedom of the flow geometry can be increased. This initial phase is equivalent to recognizing features (i)–(vi) of Section 2. An effective strategy for abbreviating this search is provided by the constructal principle, which recommends a certain (optimized) tree-shaped flow architecture. The constructal tree may be finessed (improved) by endowing its geometry with progressively more freedom to morph.

The simplest way to illustrate this behavior is to consider the flow connection between one point and a large but finite number of points. One example is the flow that bathes with minimal global resistance a disc-shaped area [6]. The stream ($\dot{m}$) emerges through the center of the disc, and exits through a large number of outlets positioned equidistantly along the disc perimeter (Fig. 3). The objective is to minimize the global flow resistance $\Delta P/\dot{m}$, where ΔP is the pressure difference between the center and the rim. The global constraints are the system size (the disc radius L) and the total volume occupied by the ducts (V). Note that the disc radius L plays the same role as L in Section 2: L is the external global constraint of the system.

Even if the search for optimal configuration starts with an assumed tree-shaped flow structure, there are still several geometric features that can be adjusted. We use the conclusions of Sections 2 and 3, and assume that every tube is straight with round cross-section and fully developed laminar flow. We also assume that local pressure drops due to junctions are negligible. The flow geometry is described by the lengths and diameters of all the ducts, the number of radial ducts (n_0) that touch the center of the disc, the distance between outlets (d, or the number of outlets on the rim, $N = 2\pi L/d$), and the number of pairing (or bifurcation) levels, which in Fig. 3 are indicated with dashed circles. In general, the number of tributaries that form a larger stream is free to vary. It was shown that the number two (dichotomy, pairing, bifurcation), which was chosen in Fig. 3, is the best such number [6,7]. Although the tree structure chosen in Fig. 3 has multiple scales and is free to change, it is already better than a huge number of other possible flow structures. In this respect, a general dichotomous tree

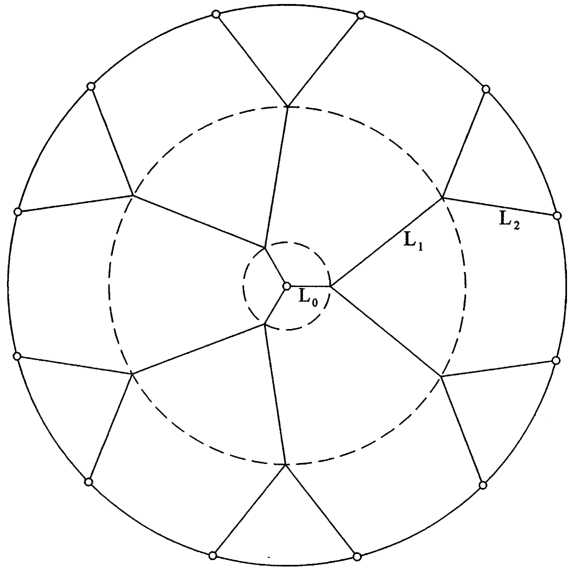

Fig. 3. The optimized tree structure with two levels of pairing and $n_0 = 3$ (or $N = 12$) [6].

structure is a near-equilibrium configuration—the equivalent of the straight duct with constant cross-section shaped as a regular polygon.

What changes in the point-to-circle tree geometry? Because L is fixed, the architecture has five degrees of freedom: the number of central tubes n_0, the length ratios L_0/L and L_1/L, and the tube diameter ratios D_1/D_0 and D_2/D_1. Another classical lesson of flow-system optimization is Murray's rule [1,8], according to which the optimal diameter ratios are all equal ($D_1/D_0 = D_2/D_1 = \cdots = 2^{-1/3}$). Murray's rule was adopted in [6], where the tree geometry was optimized for many n_0 values, and for various levels of pairing. For example, the configuration selected in Fig. 3 is the optimal flow layout for two levels of pairing and $n_0 = 3$ (or $N = 12$). It was shown that the minimized global flow resistance is represented by a formula similar to Eq. (1),

$$\frac{\Delta P}{\dot{m}} = 8\pi\nu \frac{L^3}{V^2}\hat{f} \tag{3}$$

The dimensionless factor $\hat{f}$ is a function of the number of pairing levels and N (or n_0), and is summarized in Fig. 4. The curves represent the $\hat{f}$ values minimized with respect to all the length ratios and diameter ratios.

In conclusion, the multitude of designs that depart from the optimal configuration (e.g., Fig. 3) have larger $\hat{f}$ values, and occupy the space above the respective $\hat{f}(N)$ curve. This feature is similar to what we saw in Fig. 2 for point–point flow structures. In fact, we arrive at the equivalent of Fig. 2 if we focus on only one class of flow structures: the structures with the same number of outlets on the rim, for example, $N = 48$. These structures have the same global properties (L, V). Their number of central tubes (n_0) decreases as the number of pairing levels (k) increases, namely $n_0 = 48/2^k$. This is the same as reading Fig. 4 in the vertical cut made at $N = 48$. The results are shown in Fig. 5. The performance improves as k increases. The equilibrium flow structure is the one with four levels of pairing, $k = 4$. This configuration is the most free to morph, and satisfies the additional constraint ($N = 48$). Its complexity ($k = 4$) is a result of optimization. Optimized complexity should not be confused with maximized complexity.

The continuous curve that unites the five points in Fig. 5 has the same symbolic meaning as in Fig. 2. Such a curve cannot be plotted. What exists is a conceptual demarcation line in the "performance versus freedom" domain. To the right of this line falls the 'cloud' of all the nonequilibrium flow structures that are possible. For example, the square plotted at $n_0 = 3$ and $\hat{f} = 17.6$ indicates the sub-optimal flow structure obtained more expediently by minimizing 'locally' the flow path lengths and assembling them as building blocks in a hierarchical sequence [9], instead of minimizing globally the flow resistance [6]. The edge of the cloud is concave, and the tangent to it approaches verticality near the point of equilibrium. This is another way of saying that a near-equilibrium flow structure such as the point labeled $k = 3$ performs at practically the same level as the equilibrium flow structure ($k = 4$).

The construction of Fig. 5 can be repeated for larger N values, and every new figure will be qualitatively the same as Figs. 5 and 2. The only change relative to Fig. 5 will be the increasingly larger k values of the designs that mark the edge of the cloud. The number k plays the same role as n in Fig. 2. The equilibrium flow structure

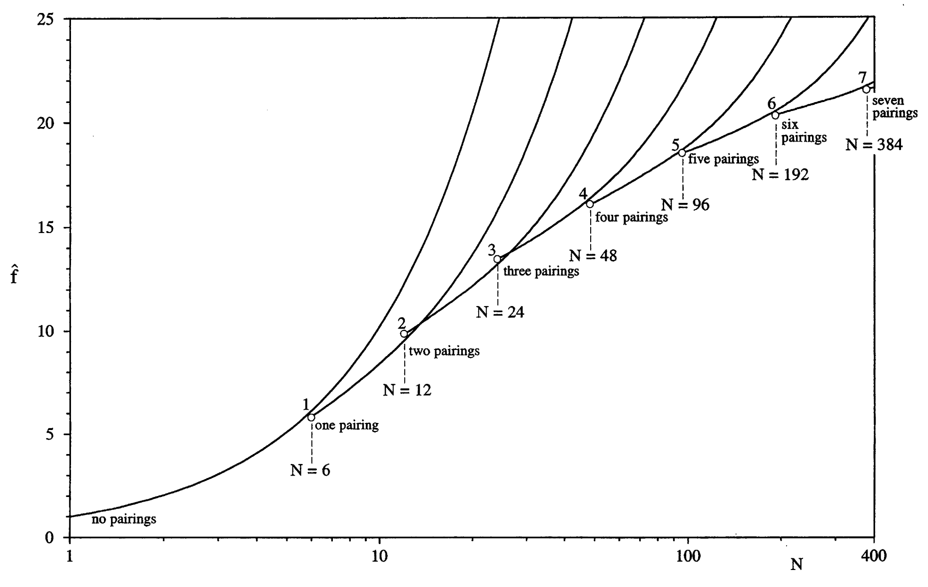

Fig. 4. The minimized global flow resistance of the tree-shaped flow structures that connect the center and the rim of a circular area [6].

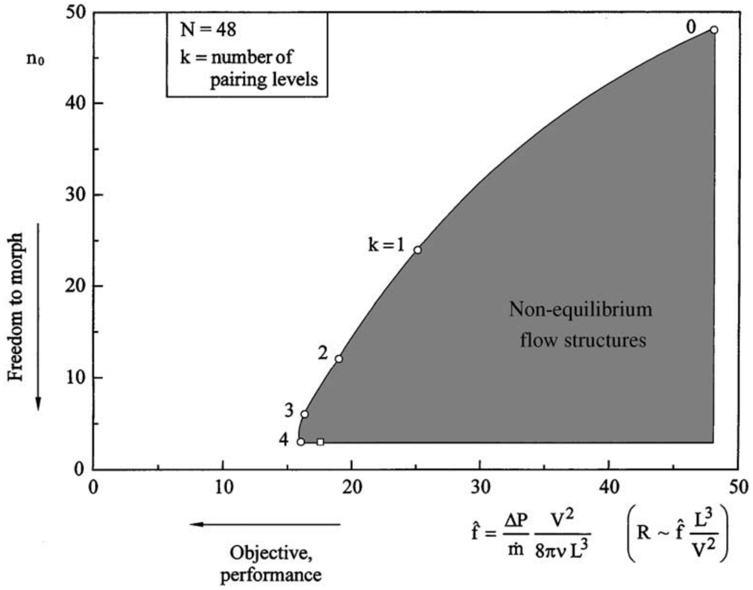

Fig. 5. The approach to the minimal global flow resistance between the center and the rim of a circular area when $N = 48$ in Fig. 4.

will be at the bottom end of the curve. It will represent the optimized architecture that results when the constant-N class is endowed with most freedom as it morphs.

5. Flow between one point and an infinite number of points

In the limit $N \to \infty$, Fig. 5 will have an infinite number of points on the edge of the cloud, and in the vicinity of the equilibrium structure the edge curve will appear continuous. The equilibrium structure will be a fractal tree, because only in this hypothetical limit it will have an infinite number of pairing levels [2,10]. This tree will connect the center of the disc with every point of the disc perimeter.

Many examples of optimized tree-shaped flow structures have been published based on constructal theory (e.g., Refs. [1,4,11–14]). They all exhibit the organization illustrated here in Figs. 2 and 5. Consider just one example of flow between an entire area and one point: conduction in a rectangular domain of area A and conductivity k_0, with uniform heat generation rate per unit area q''' (Fig. 6). The A domain is rectangular, and its boundaries are insulated except the mid point of one of the sides, which serves as heat sink ($T_{\min}$). The hot

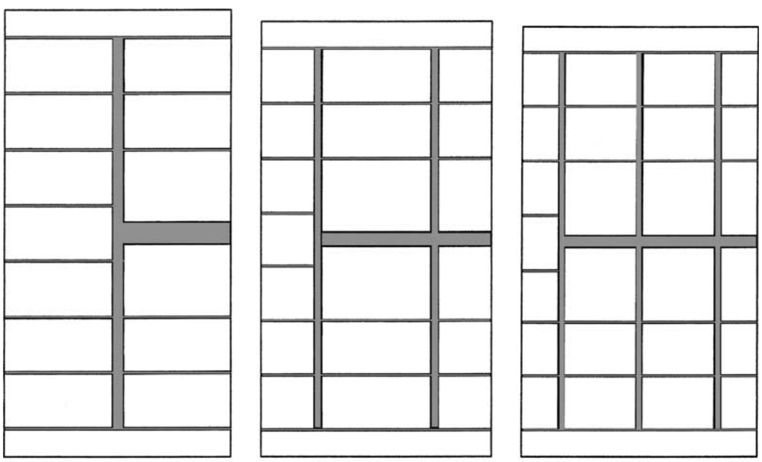

Fig. 6. Optimized high-conductivity trees embedded in low-conductivity media with uniform volumetric heat generation ($A_p/A = 0.1$, $k_p/k_0 = 300$) [14].

spot (T_{max}) occurs in one of the opposing corners of the rectangle. The flow of the global heat current $q'''A$ is aided by 'ducts' of high-conductivity material (k_p), which are distributed over A. The total area occupied by the k_p inserts is A_p.

There is an analogy between the area–point heat flow and the fluid flow structures discussed until now. The length scale $A^{1/2}$ plays the same role as the system size L. The area occupied by all the ducts (A_p) when they are projected on A plays the same role as the total duct volume V. The objective is the minimization of the global thermal resistance

$$R_t = \frac{T_{max} - T_{min}}{q'''A/k_0} \tag{4}$$

Tree structures formed by k_p-inserts were optimized numerically in Ref. [14]. One sequence of results for 'second constructs' is shown in Fig. 6. The global resistances of the three designs reported in this sequence are $R_t = 0.0379$, 0.0354 and 0.0374: the best second construct is in the middle, and has four first constructs. Each structure could still be finessed, with more degrees of freedom, en route to the equilibrium structure, which has the smallest possible global flow resistance. The nonequilibrium structures occupy a cloud of the kind shaded in Figs. 2 and 5, while the three configurations mentioned above reside near the edge of the cloud.

6. The constructal law

The flow systems discussed in Sections 2–5 have configurations that inhabit the hyperspace suggested in Fig. 7a. All the constant-L flow configurations that are possible inhabit the volume visualized by the constant-V and constant-R cuts. The bottom figure shows the view of all the possible flow structures, projected on the base plane. Plotted on the R axis is the global resistance of the flow system, namely $R = \Delta P/\dot{m}$ and R_t in the preceding examples. The abscissa accounts for the total volume occupied by the ducts (V): this is a global measure of how 'porous' or 'permeable' the flow system is. The constant-V plane that cuts through Fig. 7a is the same as the plane of Figs. 2 and 5.

The constructal law is the statement that summarizes the common observation that flow structures that survive are those that morph in one direction in time: toward configurations that make it easier for currents to flow. This holds for natural and engineered flow structures. The first such statement was [1,2]:

> For a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flow through it.

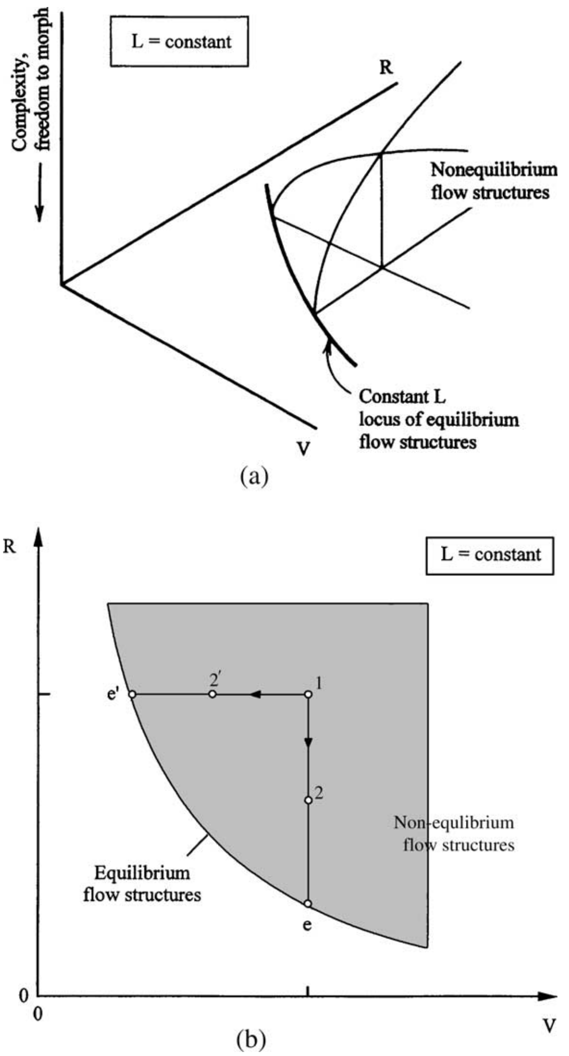

Fig. 7. The space occupied by all the flow structures when the global external size (L) is fixed.

If the flow structures are free to change (free to approach the base plane in Fig. 7a), they will move at constant-L and constant-V in the direction of progressively smaller R. If the initial configuration is represented by point 1 in Fig. 7b, then a more recent configuration is represented by point 2. The relation between the two configurations is $R_2 \leqslant R_1$ (constant L, V). If freedom to morph persists, then the flow structure will continue toward smaller R values. Any such change is characterized by

$$dR \leqslant 0 \quad (\text{constant } L, V) \tag{5}$$

The end of this migration is the equilibrium flow structure (point e), where the geometry of the flow enjoys total freedom. Equilibrium is characterized by minimal R at constant L and V. In the vicinity of the equilibrium point we have

$$\mathrm{d}R = 0 \quad \text{and} \quad \mathrm{d}^2R > 0 \quad (\text{constant } L, V) \tag{6}$$

The $R(V)$ curve shown in Fig. 7b is the edge of the cloud of possible flow architectures with the same global size L. The curve has negative slope because of the physics of flow: the flow resistance always decreases when the flow channels open up:

$$\left(\frac{\partial R}{\partial V}\right)_L < 0 \tag{7}$$

The constant-R cut through the configuration space shows another way of expressing the constructal law. If free to morph, the flow system will evolve from point 1 to point 2′ at constant L and R. In the limit of total freedom, the geometry will reach another equilibrium configuration, which is represented by point e'. The alternative analytical statement of the constructal law is

$$\mathrm{d}V \leqslant 0 \quad (\text{constant } L, R) \tag{8}$$

For changes in structure in the immediate vicinity of the equilibrium structure, we note

$$\mathrm{d}V = 0 \quad \text{and} \quad \mathrm{d}^2V > 0 \quad (\text{constant } L, R) \tag{9}$$

Paraphrasing the original statement of the constructal law, we may describe processes of type 1–2′–e' as follows:

> For a system with fixed global size and global performance to persist in time (to live), it must evolve in such a way that its flow structure occupies a smaller fraction of the available space.

The constant-V alternative to Fig. 7 is shown in Fig. 8. The lower drawing is the projection of the space of possible flow architectures on the base plane R–L. The continuous line is the locus of equilibrium flow structures at constant-V, namely the curve $R(V)$ where

$$\left(\frac{\partial R}{\partial L}\right)_V > 0 \tag{10}$$

The fact that the slope is positive is flow physics: the flow resistance always increases as the distance that must be overcome by the flow increases.

The constructal law statement can be read off Fig. 8b in two ways. One is the original statement [1,2]: at constant V and L, the evolution is from a sub-optimal structure (point 1) to one that has a lower global resistance (point 2). If the flow geometry continues to morph freely, the structure approaches the equilibrium configuration (point e). In the vicinity of point e, the changes in flow structures are characterized by Eqs. (6).

The alternative is when structural changes are made such that R remains constant while V is also fixed. Then

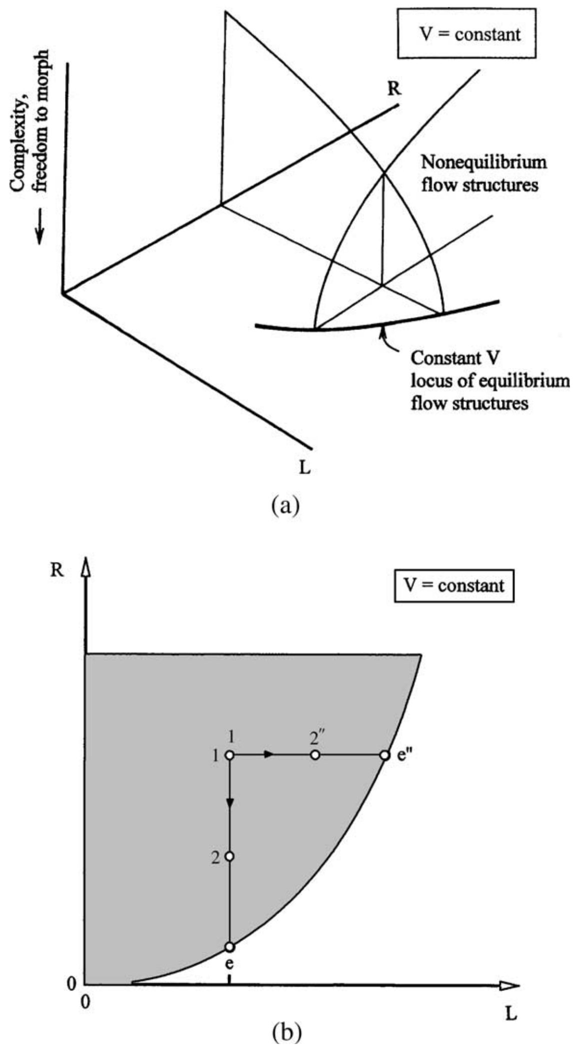

Fig. 8. The space occupied by all the flow structures when the global internal size (total duct volume V) is fixed.

the evolution in Fig. 8b is from point 1 to point 2″. Such changes mean that

$$\mathrm{d}L \geqslant 0 \quad (\text{constant } R, V) \tag{11}$$

and that the constructal law statement becomes:

> In order for a flow system with fixed global resistance (R) and internal size (V) to persist in time, the architecture must evolve in such a way that it covers a progressively larger territory.

Equilibrium is reached at point e''. The changes in flow structures in the immediate vicinity of the equilibrium structure are such that the global external dimension at equilibrium is maximal,

$$\mathrm{d}L = 0, \quad \mathrm{d}^2L < 0 \quad (\text{constant } R, V) \tag{12}$$

According to Eqs. (12), the constructal law states that the ultimate flow structure with specified global resistance (R) and internal size (V) is the largest. A flow architecture with specified R and V has a maximum size, and this global size belongs to the equilibrium architecture. A flow structure larger than this does not exist. This formulation of the constructal law has implications in natural design, e.g., the spreading of species and river deltas without access to the sea.

7. Survival by increasing efficiency, territory and compactness

It is worth examining the ground covered so far. The original statement of the constructal law was about the maximization of flow access under global size constraints (external L, internal V). This behavior is illustrated by the structural changes 1–2–e in Figs. 7b and 8b, and by Eqs. (5) and (6). This means survival by increasing efficiency—survival of the fittest. This is the physics principle behind Darwin's observations, the principle that rules not only the animate natural flow systems, but also the inanimate natural flow systems and the engineered flow systems. The engineered systems are diverse species of 'man + machine' beings.

The alternative shown by the changes 1–2″–e'' in Fig. 8b is survival by spreading: growth as the mechanism for being able to persist in time. The limit to growth is set by the specified constraints, in this case the fixed global flow resistance R and the global internal size V. A given living species (river delta, animal population) will spread over a certain, maximal territory.

An equivalent interpretation of the constructal principle is based on processes of type 1–2′–e', Fig. 7b. Flow architectures with the same performance (R) and size (L) evolve toward compactness—smaller volumes dedicated to the internal ducts, i.e., larger volumes dedicated to the working volume elements, which are the interstices. This is survival based on the maximization of the use of the available space.

8. The constructal law as an addition to thermodynamics

Changes in performance (R) can be achieved through changes of three types:

I. Flow configuration.
II. Global external size, or covered territory, L.
III. Global internal size, or duct volume, V.

The examples discussed so far showed that changes may occur in one category, or simultaneously in two or three. The simplest illustration is possible for the case of equilibrium flow architectures. For them the solid curves shown in Figs. 7b and 8b proclaim the existence of the fundamental relation $R(L, V)$, the differential of which is

$$\mathrm{d}R = Y_L\,\mathrm{d}L + Y_V\,\mathrm{d}V \quad \text{(equilibrium)} \tag{13}$$

Physics requires that the first partial derivatives of R have opposite signs, $Y_L > 0$ and $Y_V < 0$, as noted in Eqs. (10) and (7). Analytical expressions for these derivatives are available in simple cases, e.g., Eqs. (1) and (3). For example, Eq. (1) states that $R \sim L/V^2$, and from this expression one can obtain $Y_L = (\partial R/\partial L)_V$ and $Y_V = (\partial R/\partial V)_L$. Another example is the global flow resistance of a T-shaped construct of round tubes with fully developed laminar flow [7], where $R \sim A^{3/2}/V^2$, and A (or $\sim L^2$ in this paper) is the area occupied by the construct. If the flow regime is turbulent, R is proportional to $A^{7/4}/V^{5/2}$ [7]. The global resistance of an equilibrium flow structure can be decreased ($\mathrm{d}R < 0$) through changes II and III, i.e., by making the structure occupy a smaller territory ($\mathrm{d}L < 0$), and/or by endowing the structure with a larger internal flow volume ($\mathrm{d}V > 0$).

In general, when the flow architecture has not reached equilibrium, R can be decreased by means I, II and III. Then the general version of Eq. (13) is

$$\mathrm{d}R \leqslant Y_L\,\mathrm{d}L + Y_V\,\mathrm{d}V \tag{14}$$

where the inequality sign refers to the time arrow of structural changes in a flow configuration that, at least initially, was not of the equilibrium type. It is instructive to review Eqs. (5), (8) and (11), to see that Eq. (14) is a concise statement of the three analytical formulations of the constructal law that we discussed so far:

R minimum at constant L and V,
V minimum at constant R and L,
L maximum at constant V and R.

Another way to summarize the analytical formulation that we have just constructed is by recognizing the analogy between the analytical constructal law and the analytical formulation of classical thermodynamics [2]. The analogy is presented in Table 2. It is stressed further by Figs. 9 and 10, which are from present-day thermodynamics (see Ref. [2, pp. 241 and 244]). Fig. 9 expresses the energy minimum principle, which states that as the internal constraints of a closed system are removed at constant volume and entropy, the energy approaches a minimal value. Fig. 9 is analogous to Fig. 8a.

Fig. 10 is a restatement of the energy minimum principle in Helmholtz free-energy (F) representation: as the constraints of the closed system are removed at constant volume and temperature, F decreases towards a minimal value. Fig. 10 is the thermodynamics equivalent of the constructal-theory Fig. 7a.

Table 2
The concepts and principles of classical thermodynamics and constructal theory

Thermodynamics	Constructal theory
State	Flow architecture (geometry, configuration, structure)
Process, removal of internal constraints	Morphing, change in flow configuration
Properties ($U, S, \mathrm{Vol}, \ldots$)	Global objective and global constraints ($R, L, V, \ldots$)
Equilibrium state	Equilibrium flow architecture
Fundamental relation, $U(S, \mathrm{Vol}, \ldots)$	Fundamental relation, $R(L, V, \ldots)$
Constrained equilibrium states	Nonequilibrium flow architectures
Removal of constraints	Increased freedom to morph
Energy minimum principle:	Constructal principle:
U minimum at constant S and Vol	R minimum at constant L and V
Vol minimum at constant F and T	V minimum at constant R and L
S maximum at constant U and Vol	L maximum at constant V and R

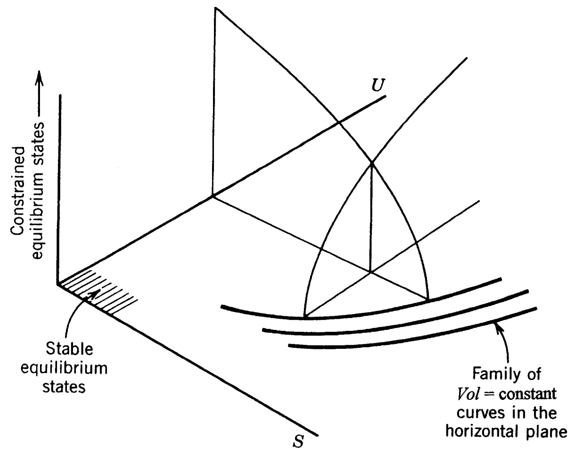

Fig. 9. The energy minimum principle (fixed volume) [2].

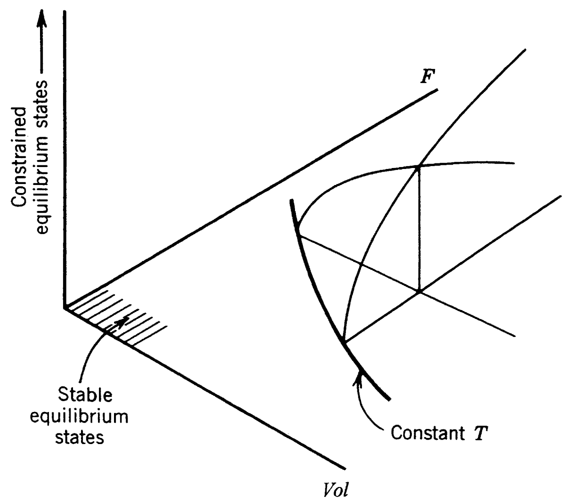

Fig. 10. The Helmholtz free-energy minimum principle (fixed temperature) [2].

9. Concluding remarks

The analytical formulation of the constructal law presented in this paper expresses a universal phenomenon: figures such as Figs. 2 and 5 characterize the evolution toward equilibrium configuration in any flow system with global objective, global constraints, and freedom to morph. In this paper we demonstrated this through examples from three wide classes of flow architectures: flow between two points, flow between a circle and its center and flow between one point and an area.

Many other examples can be contemplated, and they will all reveal the image of Figs. 2 and 5 on the road to equilibrium flow architectures. For example, if in the flow between two points the regime is turbulent (fully developed, fully rough) in every duct, then the friction factor is independent of Reynolds number. The reasoning presented in Sections 2 and 3 applies, however, instead of the group Cp^2/A of Table 1, the measure of the flow resistance of the regular-polygon cross-section is the dimensionless perimeter of the cross-section ($p/A^{1/2}$ in Table 1). The better cross-sectional shapes have lower $p/A^{1/2}$ values. It is easy to see that the "performance versus freedom" figure that can be built using the $p/A^{1/2}$ values of Table 1 will have the same features as Fig. 2. Similarly, if in the point–circle flow structures that gave us Fig. 5 we consider fully developed turbulent flow in every duct (instead of laminar flow), the emerging map in the performance versus freedom domain will be qualitatively the same as Fig. 5.

Another class of examples is based on the more realistic assumption that the pressure losses at junctions of three or more tubes are not negligible. This class of architectures can be pursued for both laminar and turbulent flow. The optimal geometric aspect ratios of flow junctions will change when the junction losses are accounted for, but the qualitative outlook of the flow architecture will not change. In the end, the world of possible flow architectures will fill a cloud with the same features as in Fig. 5.

At equilibrium the flow configuration achieves the most that its freedom to morph has to offer. Equilibrium does not mean that the flow architecture (structure, geometry, configuration) stops changing. On the contrary, it is here at equilibrium that the flow geometry enjoys most freedom to change. Equilibrium means that the global performance does not change when changes occur in the flow architecture.

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